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A model for classical spacetime coordinates

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Abstract. Field equations with general covariance are interpreted as equations for a target space describing physical spacetime co-ordinates, in terms of an underlying base space with conformal invariance. These equations admit an infinite number of inequivalent Lagrangian descriptions. A model for reparametrization invariant membranes is obtained by reversing the roles of base and target space variables in these considerations.

1. Introduction

A characteristic feature of the classical equations of general relativity is the property of general covariance; i.e that the equations are covariant under differentiable re-definitions of the spacetime co-ordinates. In the first of a series of papers investigating a class of covariant equations by Govaerts and the first author, which we called *universal field equations* [1–5] we floated the idea that these equations could be employed as a model for spacetime co-ordinates. It is one objective of this paper to explore this idea in somewhat greater depth. This is a purely classical discussion of a way of describing a co-ordinate system which is sufficiently flexible to admit the general class of functional redefinitions implied by covariance. It has nothing to do with quantum effects like the concept of a minimum compactification radius due to T duality which rules out the the notion of an infinitely precise point in spacetime. Here the discussion will remain entirely classical and will explore the idea that the spacetime co-ordinates in D dimensions may be represented by *flat* co-ordinates in $D+1$ dimensions, which transform under the conformal group in $D+1$ dimensions. There are, however, two ways to implement general covariance; one by the use of covariant derivatives, and the other by exploiting properties of determinants. In a second application the universal field equations may be regarded as describing membranes, by reversing the roles of fields and base co-ordinates. Then the covariance of fields becomes the reparametrization invariance of the new base space.

2. Multifield UFE

Suppose $X(x_i)^a$, $a = 1, \ldots, D$, $i = 1, \ldots, D+1$ denotes a set of D fields, in $D+1$ dimensional space. They may be thought of as target space co-ordinates which constitute a

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mapping from a $D+1$ dimensional base space co-odinatized by the independent variables x_i. Introduce the notation $X^a_i = \partial X^a/\partial x_i$, $X^a_{ij} = \partial^2 X^a/\partial x_i \partial x_j$. In addition, let J_k denote the Jacobian $\partial(X^a, X^b, \ldots, X^D)/\partial(x_1, \ldots, \hat{x}_k \ldots, x_{D+1})$ where x_k is the independent variable which is omitted in J_k. Now suppose that the vector field X^a satisfies the equations of motion

$$\sum_{i,k} J_i J_k X^a_{ik} = 0. \tag{2.1}$$

This is a direct generalization of the Bateman equation to D fields in $D+1$ dimensions, [1], and may be written in terms of the determinant of a bordered matrix where the diagonal blocks are of dimensions $D \times D$ and $D+1 \times D+1$ respectively as

$$\det \begin{Vmatrix} 0 & \frac{\partial X^a}{\partial x_k} \\ \frac{\partial X^b}{\partial x_j} & \sum \lambda_c \frac{\partial^2 X^c}{\partial x_j \partial x_k} \end{Vmatrix} = 0. \tag{2.2}$$

The coefficients of the arbitrary constant parameters λ_c set to zero reproduce the D equations (2.1). The solutions of these equations can be verified to possess the property that any functional redefinition of a specific solution is also a solution; i.e. the property of general covariance. A remarkable feature of (2.1) is that the equations admit infinitely many inequivalent Lagrangian formulations. Suppose $\mathcal{L}$ depends upon the fields X^a and their first derivatives X^a_j through the Jacobians subject only to the constraint that $\mathcal{L}(X^a,\ J_j)$ is a homogeneous function of the Jacobians, i.e.

$$\sum_{j=1}^{D+1} J_j \frac{\partial \mathcal{L}}{\partial J_j} = \mathcal{L}. \tag{2.3}$$

Then the Euler variation of $\mathcal{L}$ with respect to the field X^a gives

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X^a} - \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial X^a_i} & \\ &= \frac{\partial \mathcal{L}}{\partial X^a} - \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial J_j}\frac{\partial J_j}{\partial X^a_i} \\ &= \frac{\partial \mathcal{L}}{\partial X^a} - \frac{\partial^2 \mathcal{L}}{\partial X^b \partial J_j}\frac{\partial J_j}{\partial X^a_i} X^b_i - \frac{\partial \mathcal{L}}{\partial J_j}\frac{\partial^2 J_j}{\partial X^a_i \partial X^b_k} X^b_{ik} - \frac{\partial^2 \mathcal{L}}{\partial J_j \partial J_k}\frac{\partial J_j}{\partial X^a_i}\frac{\partial J_k}{\partial X^b_r} X^b_{ir}. \end{aligned} \tag{2.4}$$

The usual convention of summing over repeated indices is adhered to here. Now by the theorem of false cofactors

$$\sum_{j=1}^{D+1} \frac{\partial J_k}{\partial X^a_j} X^b_j = \delta_{ab} J_k. \tag{2.5}$$

Then, exploiting the homogeneity of $\mathcal{L}$ as a function of J_k (2.3), the first two terms in the last line of (2.4) cancel, and the term $(\partial\mathcal{L}/\partial J_j)(\partial^2 J_j/\partial X^a_i \partial X^b_k)X^b_{ik}$ vanishes by symmetry considerations. The remaining term, $(\partial^2\mathcal{L}/\partial J_j \partial J_k)(\partial J_j/\partial X^a_i)(\partial J_k/\partial X^b_r X^b_{ir})$, may be simplified as follows. Differentiation of the homogeneity equation (2.3) gives

$$\sum_{k=1}^{D+1} \frac{\partial^2 \mathcal{L}}{\partial J_j \partial J_k} J_k = 0. \tag{2.6}$$

But since $\sum_k J_k X^a_k = 0,\ \forall a$, together with symmetry, this implies that the linear equations (2.6) can be solved by

$$\frac{\partial^2 \mathcal{L}}{\partial J_i \partial J_j} = \sum_{a,b} X^a_i d^{ab} X^b_j \tag{2.7}$$

for some functions d^{ab}. Inserting this representation into (2.4) and using a similar result to (2.5);

$$\sum_{j=1}^{D+1} \frac{\partial J_j}{\partial X_k^a} X_j^b = -\delta_{ab} J_k. \tag{2.8}$$

Then, assuming $d^{a,b}$ is invertible, as is the generic case, the last term reduces to $\sum_{i,k} J_i J_k X_{ik}^a$ which, set to zero, is just the equation of motion (2.1)†.

2.1. Iteration

This procedure may be iterated; given a transformation described by the equation (2.1), from a base space of $D+2$ dimensions with co-ordinates x_i to to a target space of $D+1$ with co-ordinates Y_j which in turn are used as a base space for a similar transformation to co-ordinates X_k, $k = 1 \ldots D$ the mapping from $D+1$ dimensions to D is given in terms of the determinant of a bordered matrix of similar form to (2.2), where the diagonal blocks are of dimensions $D \times D$ and $D+2 \times D+2$ respectively;

$$\det \begin{Vmatrix} 0 & \frac{\partial X^a}{\partial x_k} \\ \frac{\partial X^b}{\partial x_j} & \sum \lambda_j \frac{\partial^2 X^j}{\partial x_j \partial x_k} \end{Vmatrix} = 0. \tag{2.9}$$

The equations which form an overdetermined set are obtained by requiring that the determinant vanishes for all choices of λ_j Further iterations yield the multifield UFE, discussed in [3], and the Lagrangian description is given by an iterative procedure.

2.2. Solutions

There are various ways to approach the question of solutions. Consider the multifield UFE;

$$\det \begin{Vmatrix} 0 & \ldots & 0 & X^1_{x_1} & \ldots & X^1_{x_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & X^n_{x_1} & \ldots & X^n_{x_d} \\ X^1_{x_1} & \ldots & X^n_{x_1} & \sum_{i=1}^n \lambda_i X^i_{x_1 x_1} & \ldots & \sum_{i=1}^n \lambda_i X^i_{x_1 x_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{x_d} & \ldots & X_{x_d} & \sum_{i=1}^n \lambda_i X^i_{x_1 x_d} & \ldots & \sum_{i=1}^n \lambda_i X^i_{x_d x_d} \end{Vmatrix} = 0 \tag{2.10}$$

where $\lambda_1, \ldots, \lambda_n$ are arbitrary constants, and the functions $X^1, \ldots, X^n$ are independent of λ_i. The equations which result from setting the coefficients of the monomials of degree $d-n$ in λ_i in the expansion of the determinant to zero form an overdetermined set, but, as we will show, this set possesses many nontrivial solutions.

The equation (2.10) may be viewed as a special case of the Monge–Ampère equation in $d+n$ dimensions, namely

$$\det \left\Vert \frac{\partial^2 u}{\partial_{y_i} \partial_{y_j}} \right\Vert_{i,j=1}^{d+n} = 0. \tag{2.11}$$

Equation (2.10) results from the restriction of u to the form

$$u(y_k) = u(x_1, \ldots, x_d, \lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i X^i \tag{2.12}$$

† This calculation without the X^a dependence of the Lagrangian can already be found in [1]; the new aspect here is the extension to include the fields themselves, following the single field example of [6].

where we have set

$$y_i = x_i \qquad i = 1, \dots, d \qquad y_{j+d} = \lambda_j \qquad j = 1, \dots, n. \tag{2.13}$$

Now the Monge–Ampère equation is equivalent to the statement that there exists a functional dependence among the first derivatives u_{y_i} of u of the form

$$F(u_{y_1}, \dots, u_{y_{d+n}}) = 0 \tag{2.14}$$

where F is an arbitrary differentiable function. Methods for the solution of this equation are known [7, 8]. Returning to the target space variables X^j, this relation becomes

$$F\left(\underbrace{\sum_{i=1}^{n} \lambda_i X^i_{x_1}}_{\omega_1}, \dots, \underbrace{\sum_{i=1}^{n} \lambda_i X^i_{x_d}}_{\omega_d}, \; X^1, \dots, X^n\right) = 0. \tag{2.15}$$

3. Exact solutions of the UFE

3.1. Implicit solutions

The general representation of a solution of this set of constraints which do not depend upon the parameters λ^i evades us; however, there are two circumstances in which a solution may be found. In the first case a class of solutions in implicit form may be obtained by taking F to be linear in the first d arguments ω_i. Then

$$F = \sum_{i=1}^{d} f_i(X^1, \dots, X^n)\omega_i = 0. \tag{3.16}$$

It can be proved that this is the generic situation for the cases of two and three fields. In general, provided there are terms linear in λ_i in F, as the X^i do not depend upon λ_i, one expects that as a minimal requirement the terms in F linear in λ_i will vanish for a solution. Equating each coefficient of λ^i in (3.16) to zero we obtain the following system of partial differential equations

$$\sum_{i=1}^{d} f_i(X^1, \dots, X^n) X^j_{x_i} = 0 \qquad j = 1, \dots, n. \tag{3.17}$$

The general solution of these equations may be represented in terms of n arbitrary smooth functions R^j, where

$$R^j(f_d x_1 - f_1 x_d, \dots, f_d x_{d-1} - f_{d-1} x_d, \; X^1, \dots, X^n) = 0. \tag{3.18}$$

The solution of these equations for X^i gives a wide class of solutions to the UFE.

3.2. Explicit solution

There is a wide class of explicit solutions to the UFE. They are simply given by choosing $X^j(x_1, \dots, x_d)$ to be a homogeneous function of x_j of weight zero, i.e.

$$\sum_{k=1}^{d} x_k \frac{\partial X^j}{\partial x_k} = 0 \qquad j = 1, \dots, n. \tag{3.19}$$

The proof of this result depends upon differentiation of (3.19) with respect to the x_i. A particularly illuminating example is the case of spherical polars; in $d = 3$, $n = 2$ take

$$X^1 = \phi = \tan^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2}}\right) \qquad X^2 = \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right). \tag{3.20}$$

Then these co-ordinates satisfy (2.9).

4. Conclusions

A wide class of solutions to the set of UFE which are generally covariant has been obtained. In order to adapt the theory to apply to possible integrable membranes, it is necessary to interchange the roles of dependent and independent variables, so that general covariance becomes reparametrization invariance of the base space [2]. In order to invert the dependent and independent variables in this fashion, it is necessary first to augment the dependent variables by some additional $d - n$ fields $Y_k(x_i)$, then consider the x_i as functions of X_j, $i = 1 \ldots n$. Although, in principle, x_i could also depend upon the artificial variables Y_k, $k = 1 \ldots d - n$, we make the restriction that this does not occur (see [2] for further details). In this case the variables x_j play the role of target space for an n-brane, dependent upon n co-ordinates X^j. Since it is fully reparametrization invariant, it may play some part in the further understanding of string theory, but this is by no means clear.

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